## Exercise 6E: Solutions

**1** **a** For all

**b** There exists

**c** For all

**d** For all

**e** There exists

**f** There exists

**g** For all

**2** **a** True

**b** False

**c** True

**d** False

**e** False

**3** **a** There exists a natural number $n$ such that $2n^2 - 4n + 31$ is not prime

**b** There exists $x \in \mathbb{R}$ such that $x^2 \leq x$

**c** For all $x \in \mathbb{R}, 2 + x^2 \neq 1 - x^2$

**d** There exists $x, y \in \mathbb{R}$ such that $(x+y)^2 \neq x^2 + y^2$

**e** For all $x, y \in \mathbb{R}, x \geq y$ implies $x^2 \geq y^2$ (Intent)

**4** **a** If we let $n = 31$ it is clear that
$2n^2 - 4n + 31 = 2 \times 31^2 - 4 \times 31 + 31$
is divisible by $31$ and so cannot be prime.

**b** Let $x = 1$ and $y = -1$ so that

$$(x+y)^2 = (1 + (-1))^2 = 0,$$

while,

$$x^2 + y^2 = 1^2 + (-1)^2 = 1 + 1 = 2,$$

**c** If $x = \frac{1}{2}$, then,

$$x^2 = \frac{1}{4} < \frac{1}{2} = x.$$

**d** If $n = 3$ then,

$$n^3 - n = 27 - 3 = 24$$

is even, although $3$ is not.

**e** If $m = n = 1$ then $m + n = 2$ while $mn = 1$.

**f** Since $6$ divides $2 \times 3 = 6$ but $6$ does not divide $2$ or $3$, the statement is false.

**5** **a** Negation: For all $n \in \mathbb{N}$, the number $9n^2 - 1$ is not a prime number.
Proof: Since

$$9n^2 - 1 = (3n-1)(3n+1),$$

and since each factor is greater than $1$, the number $9n^2 - 1$ is not a prime number.

**b** Negation: For all $n \in \mathbb{N}$, the number $n^2 + 5n + 6$ is not a prime number.
Since

$$n^2 + 5n + 6 = (n+2)(n+3),$$

and since each factor is greater than $1$, the number $9n^2 + 5n + 6$ is not a prime number.

**c** Negation: For all $x \in \mathbb{R}$, we have $2 + x^2 \neq 1 - x^2$
Proof: Suppose that $2 + x^2 = 1 - x^2$. Rearranging the equation gives,

$$\begin{aligned} 2 + x^2 &= 1 - x^2 \\ 2x^2 &= -1 \\ x^2 &= -\frac{1}{2}, \end{aligned}$$

which is impossible since $x^2 \geq 0$.

**6 a** Let $a = \sqrt{2}$ and $b = \sqrt{2}$. Then clearly each of $a$ and $b$ are irrational, although $ab = 2$ is not.

**b** Let $a = \sqrt{2}$ and $b = -\sqrt{2}$. Then clearly each of $a$ and $b$ are irrational, although $a + b = 0$ is not.

**c** Let $a = \sqrt{2}$ and $b = \sqrt{2}$. Then clearly each of $a$ and $b$ are irrational, although $\frac{a}{b} = 1$ is not.

**7 a** If $a$ is divisible by $4$ then $a = 4k$ for some $k \in \mathbb{Z}$. Therefore,

$$a^2 = (4k)^2 = 16k^2 = 4(4k^2)$$

is divisible by $4$.

**b** Converse: If $a^2$ is divisible by $4$ then $a$ is divisible by $4$.
This is clearly not true, since $2^2 = 4$ is divisible by $4$, although $2$ is not.

**8 a** If $a - b$ is divisible by $3$ then $a - b = 3k$ for some $k \in \mathbb{Z}$. Therefore,

$$a^2 - b^2 = (a-b)(a+b) = 3k(a+b)$$

is divisible by $3$.

**b** Converse: If $a^2 - b^2$ is divisible by $3$ then $a - b$ is divisible by $3$.
The converse is not true, since $2^2 - 1^2 = 3$ is divisible by $3$, although $2 - 1 = 1$ is not.

**9 a** This statement is not true since for all $a, b \in \mathbb{R}$,

$$a^2 - 2ab + b^2 = (a-b)^2 \geq 0 > -1.$$

**b** This statement is not true since for all $x \in \mathbb{R}$, we have,

$$\begin{aligned} x^2 - 4x + 5 &= x^2 - 4x + 4 - 4 + 5 \\ &= (x-2)^2 + 1 \\ &\geq 1 \\ &> \frac{3}{4}. \end{aligned}$$

**10a** The numbers can be paired as follows:

$$\begin{aligned} 16 + 9 &= 25, & 15 + 10 &= 25 \\ 14 + 11 &= 25, & 13 + 12 &= 25 \end{aligned}$$

$$1 + 8 = 9, \qquad 2 + 7 = 9,$$
$$4 + 5 = 9, \qquad 3 + 6 = 9.$$

**b** We now list each number, in descending order, with each of its potential pairs.

| | |
|---|---|
| 12 | 4 |
| 11 | 5 |
| 10 | 6 |
| 9 | 7 |
| 8 | 1 |
| 7 | 2, 9 |
| 6 | 3, 10 |
| 5 | 4 |
| 4 | 5 |
| 3 | 1, 6 |
| 2 | 7 |
| 1 | 3, 8 |

Notice that the numbers $2$ and $9$ must be paired with $7$. Therefore, one cannot pair all numbers in the required fashion.

**11** If we let $x = c$, then

$$f(c) = ac^2 + bc + c = c(ac + b + 1)$$

is divisible by $c \geq 2$.